June 2013 IJESM Volume 2, Issue 2 ISSN: 2320-0294

A GENERALIZATION OF FUZZY PRE-BOUNDARY

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Abstract

The second and third authors have recently introduced the concepts of fuzzy C – boundary [3] and fuzzy C –semi boundary [6] where C :[0, 1]→ [0, 1] is a function. The purpose of this paper is to introduce the concept of fuzzy C -pre boundary and investigate its properties in a fuzzy topological space.

MSC 2010: 54A40, 3E72.

Key words: Fuzzy C - pre boundary, fuzzy C - pre closed sets and fuzzy topology.

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1. Introduction.

The standard complement function is used to define a fuzzy closed subset of a fuzzy topological space. A fuzzy subset λ of a fuzzy topological space in the sense of Chang, is fuzzy closed if the standard complement $1-\lambda = \lambda'$ is fuzzy open. Here the standard complement is obtained by using the function $C : [0, 1] \rightarrow [0, 1]$ defined by $C(x) = 1-x$, for all $x \in [0, 1]$. Several fuzzy topologists used this type of complement while extending the concepts in general topological spaces to fuzzy topological spaces. But there are other complements in the fuzzy literature. This motivated the second and third authors to introduce the concepts of fuzzy C - closed sets [2] and fuzzy C - pre closed sets [5] in fuzzy topological spaces. In this paper, we introduce the concept of fuzzy C -pre boundary by using the arbitrary complement function C and fuzzy C -pre closure operator.

For the basic concepts and notations, one can refer Chang[7]. The concepts that are needed in this paper are discussed in the second section. The concepts of fuzzy C -pre interior and fuzzy C -pre closure are introduced in the third section. The section four is dealt with the concept of fuzzy C -pre boundary.

2. Preliminaries

Throughout this paper (X,τ) denotes a fuzzy topological space in the sense of Chang. Let $C : [0, 1] \rightarrow [0, 1]$ be a complement function. If λ is a fuzzy subset of (X,τ) then the complement $C\lambda$ of a fuzzy subset λ is defined by $C\lambda(x) = C(\lambda(x))$ for all $x \in X$. A complement function C is said to satisfy

(i) the boundary condition if $C(0) = 1$ and $C(1) = 0$,

(ii) monotonic condition if $x \leq y \Rightarrow C(x) \geq C(y)$, for all $x, y \in [0, 1]$,

(iii) involutive condition if $C(C(x)) = x$, for all $x \in [0, 1]$.

The properties of fuzzy complement function C and $C\lambda$ are given in George Klir[8] and Bageerathi et al[2]. The following lemma will be useful in sequel.

Lemma 2.1 [2]

Let $C : [0, 1] \rightarrow [0, 1]$ be a complement function that satisfies the monotonic and involutive conditions. Then for any family $\{\lambda_\alpha : \alpha \in \Delta\}$ of fuzzy subsets of X, we have

(i) $C(\sup\{\lambda_\alpha(x) : \alpha \in \Delta\}) = \inf\{C(\lambda_\alpha(x)) : \alpha \in \Delta\} = \inf\{(C\lambda_\alpha(x)) : \alpha \in \Delta\}$ and

(ii) $C(\inf\{\lambda_\alpha(x) : \alpha \in \Delta\}) = \sup\{C(\lambda_\alpha(x)) : \alpha \in \Delta\} = \sup\{(C\lambda_\alpha(x)) : \alpha \in \Delta\}$ for $x \in X$.

Definition 2.2 [2]

A fuzzy subset λ of X is fuzzy C -closed in (X,τ) if C λ is fuzzy open in (X,τ). The fuzzy C - closure of λ is defined as the intersection of all fuzzy C -closed sets μ containing λ. The fuzzy C -closure of λ is denoted by $Cl_C\,\lambda$ that is equal to $\wedge\{\mu: \mu \geq \lambda, C\,\mu\in\tau\}$.

Lemma 2.3 [2]

If the complement function C satisfies the monotonic and involutive conditions, then for any fuzzy subset λ of X,

(i) $C\,(Int\,\lambda) = Cl_C\,(C\,\lambda)$ and $C\,(Cl_C\,\lambda) = Int\,(C\,\lambda)$.

(ii) $\lambda \leq Cl_C\,\lambda$,

(iii) λ is fuzzy C -closed $\Leftrightarrow$ $Cl_C\,\lambda = \lambda$,

(iv) $Cl_C\,(Cl_C\,\lambda) = Cl_C\,\lambda$,

(v) If $\lambda \leq \mu$ then $Cl_C\,\lambda \leq Cl_C\,\mu$,

(vi) $Cl_C\,(\lambda\vee \mu) = Cl_C\,\lambda \vee Cl_C\,\mu$,

(vii) $Cl_C\,(\lambda\wedge \mu) \leq Cl_C\,\lambda \wedge Cl_C\,\mu$.

(viii) For any family $\{\lambda_\alpha\}$ of fuzzy sub sets of a fuzzy topological space we have

$$\vee\, Cl_C\,\lambda_\alpha \leq Cl_C\,(\vee\lambda_\alpha) \text{ and } Cl_C\,(\wedge\lambda_\alpha) \leq \wedge\, Cl_C\,\lambda_\alpha .$$

Lemma 2.4 [2]

Let (X,τ) be a fuzzy topological space. Let C be a complement function that satisfies the boundary, monotonic and involutive conditions. Then the following conditions hold.

(i) 0 and 1 are fuzzy C -closed sets,

(ii) arbitrary intersection of fuzzy C -closed sets is fuzzy C -closed and

(iii) finite union of fuzzy C -closed sets is fuzzy C -closed.

(iv) for any family $\{\lambda_\alpha : \alpha\in \Delta\}$ of fuzzy subsets of X. we have

$$C\,(\vee\{\lambda_\alpha: \alpha\in\Delta\}) = \wedge\{ C\,\lambda_\alpha: \alpha\in\Delta\} \text{ and } C\,(\wedge\{\lambda_\alpha: \alpha\in\Delta\}) = \vee\{ C\,\lambda_\alpha: \alpha\in\Delta\}.$$

Definition 2.5 [Definition 2.15, [3]]

A fuzzy topological space (X, τ) is C -product related to another fuzzy topological space (Y, σ) if for any fuzzy subset ν of X and ζ of Y, whenever $C\,\lambda \not\geq \nu$ and $C\,\mu \not\geq \zeta$ imply $\mathbf{C}\,\lambda \times 1 \vee 1 \times \mathbf{C}\,\mu \geq \nu \times \zeta$, where $\lambda \in \tau$ and $\mu \in\sigma$, there exist $\lambda_1 \in\tau$ and $\mu_1\in\sigma$ such that $\mathbf{C}\,\lambda_1 \geq \nu$ or $\mathbf{C}\,\mu_1 \geq \zeta$ and $\mathbf{C}\,\lambda_1 \times 1 \vee 1 \times \mathbf{C}\,\mu_1 = \mathbf{C}\,\lambda \times 1 \vee 1 \times \mathbf{C}\,\mu$.

Lemma 2.6 [Theorem 2.19, [3]]

Let (X, τ) and (Y, σ) be **C** -product related fuzzy topological spaces. Then for a fuzzy subset λ of X and a fuzzy subset μ of Y, $Cl_C(\lambda \times \mu) = Cl_C \lambda \times Cl_C \mu$.

Definition 2.7 [Definition 3.1, [4]]

Let (X,τ) be a fuzzy topological space and **C** be a complement function. Then λ is called fuzzy **C** -pre open if there exists a $\mu \in \tau$ such that $\mu \le \lambda \le Cl_C \mu$.

Lemma 2.8 [4, 5]

Let (X, τ) be a fuzzy topological space and let C be a complement function that satisfies the monotonic and involutive properties. Then a fuzzy set λ of a fuzzy topological space (X,τ) is

(i) fuzzy **C** - pre open if and only if $\lambda \le Int(Cl_C \lambda)$.

(ii) fuzzy **C** -pre closed in X if $Cl_C(Int(\lambda)) \le \lambda$.

(iii) fuzzy **C** -pre closed if and only if **C** λ is fuzzy **C** -pre open.

(iv) the arbitrary union of fuzzy **C** -pre open sets is fuzzy **C** -pre open.

Lemma 2.9 [1]

If $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the fuzzy subsets of X then

$(\lambda_1 \wedge \lambda_2) \times (\lambda_3 \wedge \lambda_4) = (\lambda_1 \times \lambda_4) \wedge (\lambda_2 \times \lambda_3)$.

Lemma 2.10 [Lemma 5.1, [2]]

Suppose f is a function from X to Y. Then $f^{-1}(\mathbf{C}\mu) = \mathbf{C}(f^{-1}(\mu))$ for any fuzzy subset μ of Y.

Definition 2.11 [7]

If λ is a fuzzy subset of X and μ is a fuzzy subset of Y, then λ × μ is a fuzzy subset of X × Y, defined by $(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(y)\}$ for each $(x, y) \in X \times Y$.

Lemma 2.12 [Lemma 2.1, [1]]

Let f : X → Y be a function. If $\{\lambda_\alpha\}$ a family of fuzzy subsets of Y, then

(i) $f^{-1}(\vee \lambda_\alpha) = \vee f^{-1}(\lambda_\alpha)$ and

(ii) $f^{-1}(\wedge \lambda_\alpha) = \wedge f^{-1}(\lambda_\alpha)$.

Lemma 2.13 [Lemma 2.2, [1]]

If λ is a fuzzy subset of X and μ is a fuzzy subset of Y, then **C** $(\lambda\times\mu)$ = **C** $\lambda \times 1 \vee 1 \times$ **C** μ.

3. Fuzzy C -pre interior and fuzzy C -pre closure

In this section, we define the concepts of fuzzy **C** -pre interior and fuzzy **C** -pre closure operators and investigate some of their basic properties.

Definition 3.1

Let (X,τ) be a fuzzy topological space and C be a complement function. Then for a fuzzy subset λ of X, the fuzzy C - pre interior of λ (briefly $pInt_C\,\lambda$), is the union of all fuzzy C - pre open sets of X contained in λ. That is,

$pInt_C(\lambda) = \vee\{\mu : \mu \le \lambda, \mu$ is fuzzy C - pre open$\}$.

Proposition 3.2

Let (X, τ) be a fuzzy topological space and let **C** be a complement function that satisfies the monotonic and involvtive conditions. Then for any fuzzy subsets λ and μ of a fuzzy topological space X, we have

(i) $pInt_C\,\lambda \le \lambda$,

(ii) λ is fuzzy C - pre open $\Leftrightarrow pInt_C\,\lambda = \lambda$,

(iii) $pInt_C(pInt_C\,\lambda) = pInt_C\,\lambda$,

(iv) If $\lambda \le \mu$ then $pInt_C\,\lambda \le pInt_C\,\mu$.

Proof.

The proof for (i) follows from Definition 3.1. Let λ be fuzzy C - pre open. Since $\lambda \le \lambda$, by using Definition 3.1, $\lambda \le pInt_C\,\lambda$. By using (i), we get $pInt_C\,\lambda = \lambda$. Conversely we assume that $pInt_C\,\lambda = \lambda$. By using Definition 3.1, λ is fuzzy C - pre open. Thus (ii) is proved. By using (ii), we get $pInt_C(pInt_C\,\lambda) = pInt_C\,\lambda$. This proves (iii). Since $\lambda \le \mu$, by using (i), $pInt_C\,\lambda \le \lambda \le \mu$. This implies that $pInt_C(pInt_C\,\lambda) \le pInt_C\,\mu$. By using (iii), we get $pInt_C\,\lambda \le pInt_C\,\mu$. This proves (iv).

Proposition 3.3

Let (X, τ) be a fuzzy topological space and let C be a complement function that satisfies the monotonic and involutive conditions. Then for any two fuzzy subsets λ and μ of a fuzzy topological space, we have (i) $pInt_C(\lambda\vee\mu) \geq pInt_C\,\lambda \vee pInt_C\,\mu$ and (ii) $pInt_C(\lambda\wedge\mu) \leq pInt_C\,\lambda \wedge pInt_C\,\mu$.

Proof.

Since $\lambda \leq \lambda \vee \mu$ and $\mu \leq \lambda\vee\mu$, by using Proposition 3.2(iv), we get $pInt_C\,\lambda \leq pInt_C(\lambda\vee\mu)$ and $pInt_C\,\mu \leq pInt_C(\lambda\vee\mu)$. This implies that $pInt_C\,\lambda \vee pInt_C\,\mu \leq pInt_C(\lambda\vee\mu)$.

Since $\lambda \wedge \mu \leq \lambda$ and $\lambda \wedge \mu \leq \mu$, by using Proposition 3.2(iv), we get $pInt_C(\lambda\wedge\mu) \leq pInt_C\,\lambda$ and $pInt_C(\lambda\wedge\mu) \leq pInt_C\,\mu$. This implies that $pInt_C(\lambda\wedge\mu) \leq pInt_C\,\lambda \wedge pInt_C\,\mu$.

Definition 3.4

Let (X,τ) be a fuzzy topological space. Then for a fuzzy subset λ of X, the fuzzy **C** - pre closure of λ (briefly $pCl_C\,\lambda$), is the intersection of all fuzzy C - pre closed sets containing λ. That is $pCl_C\,\lambda = \wedge\{\mu : \mu \geq \lambda, \mu$ is fuzzy C - pre closed$\}$.

The concepts of "fuzzy C - pre closure" and "fuzzy pre closure" are identical if C is the standard complement function.

Proposition 3.5

If the complement functions C satisfies the monotonic and involutive conditions. Then for any fuzzy subset λ of X, (i)C $(pInt_C\,\lambda) = pCl_C(C\,\lambda)$ and

(ii)C $(pCl_C\,\lambda) = pInt_C(C\,\lambda)$, where $pInt_C\,\lambda$ is the union of all fuzzy C - pre open sets contained in λ.

Proof.

By using Definition 3.1, $pInt_C\,\lambda = \vee\{\mu : \mu \leq \lambda, \mu$ is fuzzy C - pre open$\}$. Taking complement on both sides, we get C $(pInt_C(\lambda)(x))$ = C (sup$\{\mu(x) : \mu(x) \leq \lambda(x), \mu$ is fuzzy C - pre open$\}$). Since C satisfies the monotonic and involutive conditions, by using Lemma 2.1, **C** $(pInt_C(\lambda)(x))$ = inf { **C** $(\mu(x))$: $\mu(x) \leq \lambda(x)$, μ is fuzzy **C** - pre open}. This implies that **C** $(pInt_C(\lambda)(x))$ = inf { **C** $\mu(x)$): **C** $\mu(x) \geq$ **C** $\lambda(x)$, μ is fuzzy **C** - pre open}. By using Lemma 2.8, **C** μ is fuzzy **C** - pre closed, by replacing **C** μ by η, we see that **C** $(pInt_C(\lambda)(x))$ =inf$\{\eta(x) : \eta(x) \geq$ **C** $\lambda(x)$, **C** η is fuzzy **C** - pre open}. By Definition 3.4, **C** $(pInt_C(\lambda)(x)) = pCl_C$ (**C** λ) (x). This proves that **C** $(pInt_C\,\lambda) = pCl_C$ (**C** λ).

By using Definition 3.4, $pCl_C\,\lambda = \wedge\{\mu: \lambda\le\mu, \mu$ is fuzzy **C** - pre closed$\}$. Taking complement on both sides, we get **C** $(pCl_C\,\lambda\,(x))$ = **C** $(\inf\{\mu(x): \mu(x) \ge \lambda(x), \mu$ is fuzzy **C** - pre closed$\})$. Since **C** satisfies the monotonic and involutive conditions, by using Lemma 2.1, **C** $(pCl_C\,\lambda(x)) = \sup\{$**C** $(\mu(x)): \mu(x) \ge \lambda(x), \mu$ is fuzzy **C** - pre closed$\}$. That implies **C** $(pCl_C\,\lambda(x)) = \sup\{$**C** $\mu(x)$: **C** $\mu(x)) \le$ **C** $\lambda(x), \mu$ is fuzzy **C** - pre closed$\}$. By using Lemma 2.8, **C** μ is fuzzy **C** - pre open, by replacing **C** μ by η, we see that **C** $(pCl_C\,(x)) = \sup\{\eta(x): \eta(x)\le$**C** $\lambda(x), \eta$ is fuzzy **C** - pre open$\}$. By using Definition 3.1, **C** $(pCl_C\,\lambda(x)) = pInt_C\,($**C** $\lambda)(x)$. This proves **C** $(pCl_C\,(\lambda)) = pInt_C\,($**C** $\lambda)$.

Proposition 3.6

Let (X, τ) be a fuzzy topological space and let **C** be a complement function that satisfies the monotonic and involutive conditions. Then for the fuzzy subsets λ and μ of a fuzzy topological space X, we have

(i) $\lambda \le pCl_C\,\lambda$,

(ii) λ is fuzzy **C** - pre closed $\Leftrightarrow pCl_C\,\lambda = \lambda$,

(iii) $pCl_C\,(pCl_C\,\lambda) = pCl_C\,\lambda$,

(iv) If $\lambda \le \mu$ then $pCl_C\,\lambda \le pCl_C\,\mu$.

Proof.

The proof for (i) follows from $pCl_C\,\lambda= \inf\{\mu: \mu \ge \lambda, \mu$ is fuzzy **C** - pre closed$\}$.

Let λ be fuzzy **C** - pre closed. Since **C** satisfies the monotonic and involvtive conditions. Then by using Lemma 2.8, **C** λ is fuzzy **C** - pre open. By using Proposition 3.2(ii), $pInt_C\,($**C** $\lambda)$ = **C** λ. By using Proposition 3.5, **C** $(pCl_C\,\lambda)$ = **C** λ. Taking complement on both sides, we get **C** (**C** $(pCl_C\,\lambda))$ = **C** (**C** λ). Since the complement function **C** satisfies the involutive condition, $pCl_C\,\lambda = \lambda$.

Conversely, we assume that $pCl_C\,\lambda=\lambda$. Taking complement on both sides, we get **C** $(pCl_C\,\lambda)$ =**C** λ. By using Proposition 3.5, $pInt_C$**C** λ =**C** λ. By using Proposition 3.2(ii), **C** λ is fuzzy **C** - pre open. Again by using Lemma 2.8, λ is fuzzy **C** - pre closed. Thus (ii) proved.

By using Proposition 3.5, $\mathbf{C}$ ($pCl_C\ \lambda$) = $pInt_C$ ($\mathbf{C}\ \lambda$). This implies that $\mathbf{C}$ ($pCl_C\ \lambda$) is fuzzy $\mathbf{C}$ - pre open. By using Lemma 2.8, $pCl_C\ \lambda$ is fuzzy $\mathbf{C}$ - pre closed. By applying (ii), we have $pCl_C\ (pCl_C\ \lambda) = pCl_C\ \lambda$. This proves (iii).

Suppose $\lambda \le \mu$. Since $\mathbf{C}$ satisfies the monotonic condition, $\mathbf{C}\ \lambda \ge \mathbf{C}\ \mu$, that implies $pInt_C\ \mathbf{C}\ \lambda \ge pInt_C\ \mathbf{C}\mu$.Taking complement on both sides, $\mathbf{C}\ (pInt_C\ \mathbf{C}\ \lambda) \le \mathbf{C}\ (pInt_C\ \mathbf{C}\ \mu)$. Then by using Proposition 3.5, $pCl_C\ \lambda \le pCl_C\ \mu$. This proves (iv).

Proposition 3.7

Let (X, τ) be a fuzzy topological space and let $\mathbf{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for any two fuzzy subsets λ and μ of a fuzzy topological space, we have (i) $pCl_C\ (\lambda\vee\mu) = pCl_C\ \lambda \vee pCl_C\ \mu$ and
(ii) $pCl_C\ (\lambda\wedge\mu) \le pCl_C\ \lambda \wedge pCl_C\ \mu$.

Proof.

Since $\mathbf{C}$ satisfies the involutive condition, $pCl_C\ (\lambda\vee\mu) = pCl_C\ (\mathbf{C}\ (\mathbf{C}\ (\lambda\vee\mu)))$. Since $\mathbf{C}$ satisfies the monotonic and involutive conditions, by using Proposition 3.5, $pCl_C\ (\lambda\vee\mu) = \mathbf{C}\ (pInt_C\ (\mathbf{C}\ (\lambda\vee\mu)))$. Using Lemma 2.10, $pCl_C\ (\lambda\vee\mu) = \mathbf{C}\ (pInt_C\ (\mathbf{C}\ \lambda\wedge\mathbf{C}\mu))$. Again by Lemma 2.4, $pCl_C\ (\lambda\vee\mu) \le \mathbf{C}\ ((pInt_C\ \mathbf{C}\ \lambda)\wedge(pInt_C\ \mathbf{C}\ \mu)) = \mathbf{C}\ (pInt_C\ \mathbf{C}\ \lambda)\vee\mathbf{C}\ (pInt_C\mathbf{C}\mu)$. By using Proposition 3.5, $pCl_C\ (\lambda\vee\mu) \le sCl_C\ \lambda\vee pCl_C\ \mu$. Also $pCl_C\ (\lambda) \le pCl_C\ (\lambda\vee\mu)$ and $pCl_C\ (\mu) \le pCl_C\ (\lambda\vee\mu)$ that implies $pCl_C\ (\lambda)\vee pCl_C\ (\mu) \le pCl_C\ (\lambda\vee\mu)$. It follows that $pCl_C\ (\lambda\vee\mu) = pCl_C\ \lambda\vee pCl_C\ \mu$.

Since $pCl_C\ (\lambda\wedge\mu) \le pCl_C\ \lambda$ and $pCl_C\ (\lambda\wedge\mu) \le pCl_C\ \mu$, it follows that $pCl_C\ (\lambda\wedge\mu) \le pCl_C\ \lambda \wedge pCl_C\ \mu$.

Proposition 3.8

Let $\mathbf{C}$ be a complement function that satisfies the monotonic and involutive conditions. Then for any family $\{\lambda_\alpha\}$ of fuzzy subsets of a fuzzy topological space, we have (i) $\vee(pCl_C\ \lambda_\alpha) \le pCl_C\ (\vee\lambda_\alpha)$ and (ii) $pCl_C\ (\wedge\lambda_\alpha) \le \wedge(pCl_C\ \lambda_\alpha)$

Proof.

For every β, $\lambda_\beta \le \vee\lambda_\alpha \le pCl_C\ (\vee\lambda_\alpha)$. By using Proposition 3.6(iv), $pCl_C\ \lambda_\beta \le pCl_C\ (\vee\lambda_\alpha)$ for every β. This implies that $\vee pCl_C\ \lambda_\beta \le pCl_C\ (\vee\lambda_\alpha)$. This proves (i). Now

$\wedge\lambda_\alpha \leq \lambda_\beta$ for every β. Again using Proposition 3.6(iv), we get $pCl_C(\wedge\lambda_\alpha) \leq pCl_C\,\lambda_\beta$. This implies that $pCl_C(\wedge\lambda_\alpha) \leq \wedge pCl_C\,\lambda_\alpha$. This proves (ii).

4. Fuzzy C -pre boundary

In this section, the concept of fuzzy **C** -pre boundary is introduced and its properties are discussed.

Definition 4.1

Let λ be a fuzzy subset of a fuzzy topological space X and let **C** be a complement function. Then the fuzzy **C** - pre boundary of λ is defined as

$pBd_C\,\lambda = pCl_C\,\lambda \wedge pCl_C\,(\mathbf{C}\,\lambda)$.

Since the arbitrary intersection of fuzzy **C** -pre closed sets is fuzzy **C** -pre closed, $pBd_C\,\lambda$ is fuzzy **C** - pre closed.

We identify $pBd_C\,\lambda$ with pBd(λ) when **C** (x) = 1–x, the usual complement function.

Proposition 4.2

Let (X,τ) be a fuzzy topological space and **C** be a complement function that satisfies the involutive condition. Then for any fuzzy subset λ of X, $pBd_C\,\lambda = pBd_C\,(\mathbf{C}\,\lambda)$.

Proof.

By using Definition 4.1, $pBd_C\,\lambda = pCl_C\,\lambda \wedge pCl_C\,(\mathbf{C}\,\lambda)$. Since **C** satisfies the involutive condition **C** (**C** λ) = λ, that implies $pBd_C\,\lambda = pCl_C\,(\mathbf{C}\,\lambda) \wedge pCl_C\,\mathbf{C}\,(\mathbf{C}\,\lambda)$. Again by using Definition 4.1, $pBd_C\,\lambda = pBd_C\,(\mathbf{C}\,\lambda)$.

The following example shows that, the word "involutive" can not be dropped from the hypothesis of Proposition 4.2.

Example 4.3

Let X = {a, b, c} and τ = {0, {$a_{.3}$, $b_{.7}$}, {$a_{.5}$, $b_{.2}$, $c_{.6}$}, {$a_{.5}$, $b_{.7}$, $c_{.6}$}, {$a_{.3}$, $b_{.2}$}, 1}.

Let **C** (x) = $\frac{1-x}{1+x^2}$, 0 ≤ x ≤ 1, be the complement function. We note that the complement function **C** does not satisfy the involutive condition. The family of all fuzzy **C** -closed sets is **C**(τ)={0,{$a_{.642}$, $b_{.201}$, c_1},{$a_{.4}$,$b_{.769}$, $c_{.294}$},{$a_{.4}$, $b_{.201}$, $c_{.294}$},{$a_{.642}$, $b_{.769}$,c_1}, 1}.

Let λ = {$a_{.5}$, $b_{.8}$, $c_{.4}$}. Then it can be calculated that $pCl_C\,\lambda$ = {$a_{.5}$, $b_{.8}$, $c_{.4}$}.

Now **C** λ = {$a_{.4}$, $b_{.122}$, $c_{.57}$} and the value of pCl_C **C** λ = {$a_{.4}$, $b_{.122}$, $c_{.517}$}. Hence $pBd_C\,\lambda = pCl_C\lambda \wedge pCl_C\,(\mathbf{C}\,\lambda)$ = {$a_{.4}$, $b_{.122}$, $c_{.517}$}. Also **C** (**C** λ) = {$a_{.517}$, $b_{.865}$, $c_{.381}$}, p$Cl_{\mathbf{C}}$**C** (**C** λ) = 1. pBd$_{\mathbf{C}}$ **C** λ = p$Cl_{\mathbf{C}}$ **C** λ ∧ p$Cl_{\mathbf{C}}$ **C** (**C** λ) = {a.$_4$, b.$_{122}$, c.$_{381}$}. This implies that pBd$_{\mathbf{C}}$ λ ≠ pBd$_{\mathbf{C}}$ **C** λ.

Proposition 4.4

Let (X,τ) be a fuzzy topological space and **C** be a complement function that satisfies the monotonic and involutive conditions. If λ is fuzzy **C** - pre closed, then pBd$_{\mathbf{C}}$ λ ≤ λ.

Proof.

Let λ be fuzzy **C** - pre closed. By using Definition 4.1, pBd$_{\mathbf{C}}$ λ = p$Cl_{\mathbf{C}}$ λ∧p$Cl_{\mathbf{C}}$ (**C** λ). Since **C** satisfies the monotonic and involutive conditions, by using Proposition 3.6(ii), we have p$Cl_{\mathbf{C}}$ λ = λ. Hence pBd$_{\mathbf{C}}$ λ ≤ p$Cl_{\mathbf{C}}$ λ = λ.

The following example shows that if the complement function **C** does not satisfy the monotonic and involutive conditions, then the conclusion of Proposition 4.4 is false.

Example 4.5

Let X = {a, b, c} and τ = **{**0, {a.$_6$, b.$_9$}, {a.$_7$, b.$_3$}, {a.$_6$, b.$_3$}, {a.$_7$, b.$_9$}, 1**}**.

Let **C** (x) = $\frac{2x}{1+x}$, 0 ≤ x ≤ 1, be a complement function. From this, we see that the complement function **C** does not satisfy the monotonic and involutive conditions. The family of all fuzzy **C** - closed sets is given by **C** (τ) = **{**0, {a.$_{75}$, b.$_{947}$}, {a.$_{824}$, b.$_{462}$}, {a.$_{75}$,b.$_{462}$}, {a.$_{824}$,b.$_{947}$}, 1**}**. Let λ = {a.$_8$, b.$_3$}, it can be found that $Cl_{\mathbf{C}}$ λ = {a.$_{824}$, b.$_{462}$} and *Int* $Cl_{\mathbf{C}}$ λ = {a.$_7$, b.$_3$}. That implies *Int* $Cl_{\mathbf{C}}$ λ ≤ λ. This shows that λ is fuzzy **C** - pre closed. Further it can be calculated that p$Cl_{\mathbf{C}}$ λ = {a.$_{85}$, b.$_{632}$}. Now **C** λ = {a.$_{889}$, b.$_{67}$} and p$Cl_{\mathbf{C}}$ **C** λ = {a.$_{889}$, b.$_{67}$}. Hence pBd$_{\mathbf{C}}$ λ = p$Cl_{\mathbf{C}}$ λ ∧ p$Cl_{\mathbf{C}}$ (**C** λ) = {a.$_{85}$, b.$_{632}$}. This implies that pBd$_{\mathbf{C}}$ λ ≰ λ. This shows that the conclusion of Proposition 4.4 is false.

Proposition 4.6

Let (X,τ) be a fuzzy topological space and **C** be a complement function that satisfies the monotonic and involutive conditions. If λ is fuzzy **C** -pre open then pBd$_{\mathbf{C}}$ λ ≤ **C** λ.

Proof.

Let λ be fuzzy **C** -pre open. Since **C** satisfies the involutive condition, this implies that **C** (**C** λ) is fuzzy **C** -pre open. By using Lemma 2.8, **C** λ is fuzzy **C** - pre closed.

Since **C** satisfies the monotonic and the involutive conditions, by using Proposition 4.4, pBd_C (**C** λ) ≤ **C** λ. Also by using Proposition 4.2, we get pBd_C (λ)≤**C** λ. This completes the proof.

Example 4.7

Let X = {a, b, c} and τ = **{**0, {$a_{.3}$, $b_{.7}$}, {$a_{.5}$, $b_{.2}$, $c_{.6}$}, {$a_{.5}$, $b_{.7}$, $c_{.6}$}, {$a_{.3}$, $b_{.2}$}, 1**}**.

Let **C** (x) = $\frac{1-x}{1+x^2}$, 0 ≤ x ≤ 1, be the complement function. We note that the complement function **C** does not satisfy the involutive condition. The family of all fuzzy **C** -closed sets is **C**(τ)=**{**0,{$a_{.642}$, $b_{.201}$, c_1},{$a_{.4}$,$b_{.769}$, $c_{.294}$},{$a_{.4}$, $b_{.201}$, $c_{.294}$},{$a_{.642}$, $b_{.769}$,c_1}, 1**}**.

Let λ = {$a_{.4}$, $b_{.122}$, $c_{.57}$}, the value of pCl_C λ = {$a_{.4}$, $b_{.122}$, $c_{.517}$} and **C** λ = {$a_{.517}$, $b_{.865}$, $c_{.381}$}, it follows that pBd_C λ = pCl_Cλ∧pCl_C (**C** λ) = {$a_{.4}$, $b_{.122}$, $c_{.381}$}. This shows that pBd_C λ ≰ **C** λ. Therefore the conclusion of Proposition 4.6 is false.

Proposition 4.8

Let (X, τ) be a fuzzy topological space and **C** be a complement function that satisfies the monotonic and involutive conditions. If λ ≤ μ and μ is fuzzy **C** -pre closed then pBd_C λ ≤ μ.

Proof.

Let λ ≤ μ and μ be fuzzy **C** - pre closed. Since **C** satisfies the monotonic and involutive conditions, by using Proposition 3.6(iv), we have λ ≤ μ implies pCl_C λ ≤ pCl_C μ. By using Definition 4.1, pBd_C λ = pCl_C λ ∧ pCl_C (Cλ). Since pCl_C λ ≤ pCl_C μ, we have pBd_C λ ≤ pCl_C μ∧pCl_C (**C** λ) ≤ pCl_C μ. Again by using Proposition 3.6 (ii), we have pCl_C μ = μ. This implies that pBd_C λ ≤ μ.

The following example shows that if the complement function **C** does not satisfy the monotonic and involutive conditions, then the conclusion of Proposition 4.8 is false.

Example 4.9

From Example 4.5, let X = {a, b} and τ = **{**0, {$a_{.6}$, $b_{.9}$}, {$a_{.7}$, $b_{.3}$}, {$a_{.6}$, $b_{.3}$}, {$a_{.7}$, $b_{.9}$}, 1**}**. Let λ = {$a_{.7}$, $b_{.45}$} and μ = {$a_{.76}$, $b_{.5}$}. Then it can be found that *Int* μ = {$a_{.7}$, $b_{.3}$} and Cl_C *Int* μ = {$a_{.75}$, $b_{.462}$}. That implies Cl_C *Int* μ ≤ μ. This show that μ is fuzzy **C** -pre closed. It can be computed that pCl_C λ = {$a_{.8}$, $b_{.47}$}. Now **C**λ = {$a_{.824}$, $b_{.62}$} and pCl_C **C** λ = {$a_{.824}$, $b_{.47}$}. pBd_C λ = pCl_C λ ∧ pCl_C (**C** λ) = { $a_{.8}$, $b_{.47}$}. This shows that pBd_C λ ≰ μ. Therefore the conclusion of Proposition 4.8 is false.

Proposition 4.10

Let (X,τ) be a fuzzy topological space and **C** be a complement function that satisfies the monotonic and involutive conditions. If $\lambda \leq \mu$ and μ is fuzzy **C** - pre open then $pBd_{\mathbf{C}}\,\lambda \leq \mathbf{C}\,\mu$.

Proof.

Let $\lambda \leq \mu$ and μ is fuzzy **C** - pre open. Since **C** satisfies the monotonic condition, by using Proposition 3.6(iv), we have $C\,\mu \leq C\,\lambda$ that implies $pCl_C C\,\mu \leq pCl_C\,C\,\lambda$. By using Definition 4.1, $pBd_C\,\lambda = pCl_C\,\lambda \wedge pCl_C\,C\,\lambda$. Taking complement on both sides, we get **C** $(pBd_C\,\lambda) = C\,(pCl_C\,\lambda \wedge pCl_C\,(C\,\lambda))$. Since **C** satisfies the monotonic and involutive conditions, by using Lemma 2.1, we have $C\,(pBd_C\,\lambda) = C\,(pCl_C\,\lambda) \vee C\,(pCl_{\mathbf{C}}\,(C\,\lambda))$. Since $pCl_C\,C\,\mu \leq pCl_C\,C\,\lambda$, $C\,(pBd_C\,\lambda) \geq C\,(pCl_C\,C\,\mu) \vee C\,(pCl_{\mathbf{C}}\,\lambda)$, by Proposition 3.5(ii), $C\,(pBd_{\mathbf{C}}\,\lambda) \geq pInt_C\,\mu \vee pInt_C\,C\,\lambda \geq pInt_{\mathbf{C}}\,\mu$. Since μ is fuzzy **C** - pre open, **C** $(pBd_{\mathbf{C}}\,\lambda) \geq \mu$. Since **C** satisfies the monotonic conditions, $pBd_{\mathbf{C}}\,\lambda \leq \mathbf{C}\,\mu$.

The following example shows that if the complement function **C** does not satisfy the monotonic and involutive conditions, then the conclusion of Proposition 4.10 is false.

Example 4.11

From Example 4.5, let $X = \{a, b\}$ and $\tau = \{0, \{a_{.6}, b_{.9}\}, \{a_{.7}, b_{.3}\}, \{a_{.6}, b_{.3}\}, \{a_{.7}, b_{.9}\}, 1\}$. Let $\lambda = \{a_{.6}, b_{.3}\}$ and $\mu = \{a_{.65}, b_{.4}\}$. Then it can be evaluated that $Int\,\lambda = \{a_{.6}, b_{.3}\}$ and $Cl_C\,Int\,\lambda = \{a_{.75}, b_{.462}\}$. Thus we see that $\lambda \leq Cl_C\,(Int\lambda)$. By using Lemma 2.8, λ is fuzzy C - pre open. It can be computed that $pCl_C\,\lambda = \{a_{.85}, b_{.632}\}$. Now $C\,\lambda = \{a_{.75}, b_{.462}\}$ and $pCl_C\,C\,\lambda = \{a_{.85}, b_{.632}\}$. $pBd_C\,\lambda = pCl_C\,\lambda \wedge pCl_C\,(C\,\lambda) = \{a_{.85}, b_{.632}\}$. This shows that $pBd_C\,\lambda \not\leq C\,\mu$.

Proposition 4.12

Let (X,τ) be a fuzzy topological space. Let C be a complement function that satisfies the monotonic and involutive conditions. Then for any fuzzy subset λ of X, we have **C** $(pBd_C\,\lambda) = pInt_C\lambda \vee pInt_C\,(C\,\lambda)$.

Proof.

By using Definition 4.1, $pBd_C\,\lambda = pCl_C\,\lambda \wedge pCl_C\,(C\,\lambda)$. Taking complement on both sdes, we get $C\,(pBd_C\,\lambda) = C\,(pCl_C\,\lambda \wedge pCl_C\,(C\,\lambda))$. Since **C** satisfies the monotonic and involutive conditions, by using Lemma 2.4(ii), $C\,(pBd_C\,\lambda) = C\,(pCl_C\,\lambda) \vee C$

(pCl_C (C λ)). Also by using Proposition 3.6(ii), that implies C (pBd_C λ) = pInt_C (C λ)∨pInt_C (C (C λ)). Since C satisfies the involutive condition, C (pBd_Cλ) = pInt_C λ∨pInt_C (C λ).

The following example shows that if the monotonic and involutive conditions of the complement function **C** are dropped , then the conclusion of Proposition 4.12 is false.

Example 4.13

Let X = {a, b} and τ = **{**0 , {$a_{.3}$, $b_{.8}$}, {$a_{.2}$, $b_{.5}$}, {$a_{.7}$, $b_{.1}$}, {$a_{.3}$, $b_{.5}$}, {$a_{.3}$, $b_{.1}$}, {$a_{.2}$, $b_{.1}$}, {$a_{.7}$, $b_{.8}$}, {$a_{.7}$, $b_{.5}$}, 1**}**. Let C (x) = $\sqrt{x}$, 0 ≤ x ≤ 1 be the complement function. From this example, we see that C does not satisfy the monotonic and involutive conditions. The family of all fuzzy **C** -closed sets is C (τ) = **{**0, {$a_{.548}$, $b_{.894}$}, {$a_{.447}$, $b_{.707}$},{$a_{.837}$, $b_{.316}$}, {$_{.548}$, $b_{.707}$},{$a_{.548}$, $b_{.316}$}, {$a_{.447}$, $b_{.316}$},{$a_{.837}$, $b_{.894}$}, {$a_{.837}$, $b_{.707}$},1**}**.

Let λ = {$a_{.6}$, $b_{.3}$}. Then it can be evaluated that pInt_C λ = {$a_{.3}$, $b_{.1}$}, C λ = {$a_{.775}$, $b_{.548}$} and pInt_C C λ = {$a_{.7}$, $b_{.5}$}. Thus we see that pInt_C λ ∨ pInt_C C λ = {$a_{.775}$, $b_{.548}$}. It can be computed that pCl_C λ = {$a_{.5}$, $b_{.8}$}. Now C λ = {$a_{.775}$, $b_{.548}$}, pCl_C C λ = {$a_{.837}$, $b_{.707}$} and pBd_C λ = pCl_C λ ∧ pCl_C (C λ) = {$a_{.5}$, $b_{.707}$}. Also C (pBd_C λ) = {$a_{.707}$, $b_{.840}$}. Thus we see that **C** (pBd_C λ) ≠ sInt_C λ ∨ sInt_C C λ. Therefore the conclusion of Proposition 4.12 is false.

Proposition 4.14

Let (X,τ) be a fuzzy topological space. Let C be a complement function that satisfies the monotonic and involutive conditions. Then for any fuzzy subset λ of X, we have pBd_C (λ) = pCl_C (λ)∧C (pInt_C (λ)).

Proof.

By using Definition 4.1, we have pBd_C (λ) = pCl_C (λ)∧pCl_C (C λ). Since C satisfies the monotonic and involutive conditions, by using Proposition 3.5(ii), we have pBd_C (λ) = pCl_C (λ)∧C (pInt_C (λ)).

The next example shows that if the complement function C does not satisfy the monotonic and involutive conditions, then the conclusion of Proposition 4.14 is false.

Example 4.15

From Example 4.5, let X = {a, b} and τ = {0, {a.$_6$, b.$_9$}, {a.$_7$, b.$_3$}, {a.$_6$, b.$_3$}, {a.$_7$, b.$_9$}, 1}. Let λ = {a.$_9$, b.$_5$}. Then it can be evaluated that pInt_C λ = {a.$_{75}$, b.$_{462}$} and C (pInt_C λ) = {a.$_{857}$, b.$_{632}$} and it can be computed that pCl_C λ = {a.$_9$, b.$_5$}. Now C λ = {a.$_{947}$, b.$_{67}$}, pCl_C C λ = {a.$_{947}$, b.$_{67}$} and pBd$_C$ λ = pCl_C $\lambda \wedge$ pCl_C (C λ) = {a.$_9$, b.$_5$}. Also pCl_C $\lambda \wedge$ C (pInt_C λ) = {a.$_{857}$, b.$_5$}. Thus we see that pBd$_C$ $\lambda \neq$ pCl_C $\lambda \wedge$ C (pInt_C λ). Therefore the conclusion of Proposition 4.14 is false.

Proposition 4.16

Let (X,τ) be a fuzzy topological space. Let C be a complement function that satisfies the monotonic and involutive conditions. Then for any subset λ of X, pBd$_C$ (pInt_C (λ)) $\leq$ pBd$_C$ (λ).

Proof.

Since C satisfies the monotonic and involutive conditions, by using Proposition 4.14, we have pBd$_C$ (pInt_C (λ)) = pCl_C (pInt_C (λ))$\wedge$C (pInt_C (pInt_C (λ))). Since pInt_C (λ) is fuzzy C -pre open, pBd$_C$ (pInt_C (λ)) = pCl_C (pInt_C (λ))$\wedge$**C** (pInt_C (λ)). Since pInt_C (λ) $\leq \lambda$, by using Proposition 3.6(ii), pCl_C (pInt_C (λ)) $\leq$ pCl_C (λ). Thus pBd$_C$ (pInt_C (λ)) $\leq$ pCl_C (λ)$\wedge$C (pInt_C (λ)). Since C satisfies the monotonic and involutive conditions, by using Proposition 3.5, pBd$_C$ (pInt_C (λ)) $\leq$ pCl_C (λ)$\wedge$pCl_C (C λ). By using Definition 4.1, we have pBd$_C$ (sInt_C (λ)) $\leq$ pBd$_C$ (λ).

Proposition 4.17

Let (X,τ) be a fuzzy topological space. Let C be a complement function that satisfies the monotonic and involutive conditions. Then pBd$_C$ (pCl_C (λ)) $\leq$ pBd$_C$ (λ).

Proof.

Since **C** satisfies the monotonic and involutive conditions, by using Proposition 4.14, pBd$_C$ (pCl_C (λ)) = pCl_C (pCl_C (λ)) $\wedge$ C (pInt_C (pCl_C (λ))). By using Proposition 3.6(iii), we have pCl_C (pCl_C (λ)) = pCl_C (λ), that implies pBd$_C$ (pCl_C (λ)) = pCl_C (λ)$\wedge$C (pInt_C (pCl_C (λ))). Since $\lambda \leq$ pCl_C (λ), that implies pInt_C (λ) $\leq$ pInt_C (Cl_C (λ)). Therefore, pBd$_C$ (pCl_C (λ)) $\leq$ pCl_C (λ)$\wedge$C (pInt_C (λ)). By using Proposition 3.5 (ii), and by using Definition 4.1, we get pBd$_C$ (pCl_C (λ)) $\leq$ pBd$_C$ (λ).

Theorem 4.18

Let (X,τ) be a fuzzy topological space. Let C be a complement function that satisfies the monotonic and involutive conditions. Then $pBd_C(\lambda\vee\mu)\leq pBd_C\,\lambda\vee pBd_C\,\mu$.

Proof.

By using Definition 4.1, $pBd_C(\lambda\vee\mu) = pCl_C(\lambda\vee\mu) \wedge pCl_C(C(\lambda\vee\mu))$. Since C satisfies the monotonic and involutive conditions, by using Proposition 3.7(i), that implies $pBd_C(\lambda\vee\mu) = (pCl_C(\lambda)\vee pCl_C(\mu))\wedge pCl_C(C(\lambda\vee\mu))$. By using Lemma 2.4 and Proposition 3.7(ii), $pBd_C(\lambda\vee\mu) \leq (pCl_C(\lambda) \vee pCl_C(\mu)) \wedge (pCl_C(C\,\lambda) \wedge pCl_C(C\,\mu))$. That is, $pBd_C(\lambda\vee\mu) \leq (pCl_C(\lambda) \wedge pCl_C(C\,\lambda))\vee(pCl_C(\mu))\wedge pCl_C(C\,\mu))$. Again by using Definition 4.1, $pBd_C(\lambda\vee\mu) \leq pBd_C(\lambda) \vee pBd_C(\mu)$.

Theorem 4.19

Let (X,τ) be a fuzzy topological space. Suppose the complement function C satisfies the monotonic and involutive conditions. Then for any two fuzzy subsets λ and μ of a fuzzy topological space X, we have

$pBd_C(\lambda\wedge\mu) \leq (pBd_C(\lambda)\wedge pCl_C(\mu)) \vee (pBd_C(\mu)\wedge pCl_C(\lambda))$.

Proof.

By using Definition 4.1, we have $pBd_C(\lambda\wedge\mu) = pCl_C(\lambda\wedge\mu) \wedge pCl_C(C(\lambda\wedge\mu))$. Since C satisfies the monotonic and involutive conditions, by using Proposition 3.7(i), Proposition 3.7 (ii) and by using Lemma 2.4(iv), we get $pBd_C(\lambda\wedge\mu) \leq (pCl_C(\lambda)\wedge pCl_C(\mu))\wedge (pCl_C(C\,\lambda)\vee pCl_C(C\,\mu))$ is equal to $[pCl_C(\lambda)\wedge pCl_C(C\,\lambda)] \wedge(pCl_C(\mu)) \vee [pCl_C(\mu)\wedge pCl_C(C\,\mu)] \wedge pCl_C(\lambda)$. Again by Definition 4.1, we get $pBd_C(\lambda\wedge\mu) \leq (pBd_C(\lambda)\wedge pCl_C(\mu))\vee(pBd_C(\mu)\wedge pCl_C(\lambda))$.

Proposition 4.20

Let (X, τ) be a fuzzy topological space. Suppose the complement function C satisfies the monotonic and involutive conditions. Then for any fuzzy subset λ of a fuzzy topological space X, we have (i) $pBd_C(pBd_C(\lambda)) \leq pBd_C(\lambda)$

(ii) $pBd_C\, pBd_C\, pBd_C\,\lambda \leq pBd_C\, pBd_C\,\lambda$.

Proof.

By using Definition 4.1, $pBd_C\,\lambda = pCl_C\,\lambda \wedge pCl_C(C\,\lambda)$. We have

$pBd_C\, pBd_C\,\lambda = pCl_C(pBd_C\,\lambda) \wedge pCl_C[C(pBd_C\,\lambda)] \leq pCl_C(pBd_C\,\lambda)$. Since **C** satisfies the monotonic and involutive conditions, by using Proposition 3.6(ii), $pCl_C\,\lambda= \lambda$, where λ is fuzzy C

-pre closed. Here $pBd_C\,\lambda$ is fuzzy C -pre closed. So, $pCl_C(pBd_C\,\lambda) = pBd_C\,\lambda$. This implies that $pBd_C\,pBd_C\,\lambda \leq pBd_C\,\lambda$. This proves (i).

(ii) Follows from (i).

Proposition 4.21

Let λ be a fuzzy C -pre closed subset of a fuzzy topological space X and μ be a fuzzy C -pre closed subset of a fuzzy topological space Y, then $\lambda \times \mu$ is a fuzzy C -pre closed set of the fuzzy product space $X \times Y$ where the complement function C satisfies the monotonic and involutive conditions.

Proof.

Let λ be a fuzzy C - pre closed subset of a fuzzy topological space X. Then by applying Lemma 2.8, C λ is fuzzy C - pre open set in X. Also if C λ is fuzzy C - pre open set in X, then C $\lambda \times 1$ is fuzzy C - pre open in $X \times Y$. Similarly let μ be a fuzzy C - pre closed subset of a fuzzy topological space X. Then by using Lemma 2.8, C μ is fuzzy C - pre open set in Y. Also if C μ is fuzzy C - pre open set in Y then C $\mu \times 1$ is fuzzy C - pre open in $X \times Y$. Since the arbitrary union of fuzzy C - pre open sets is fuzzy C -pre open. So, C $\lambda \times 1 \vee 1 \times$ C μ is fuzzy C - pre open in $X \times Y$. By using Lemma 2.13, C $(\lambda \times \mu)$ = C $\lambda \times 1 \vee 1 \times$ C μ, hence **C** $(\lambda \times \mu)$ is fuzzy C - pre open. By using Lemma 2.8, $\lambda \times \mu$ is fuzzy C - pre closed of the fuzzy product space $X \times Y$.

Theorem 4.22

Let C be a complement function that satisfies the monotonic and involutive conditions. If λ is a fuzzy subset of a fuzzy topological space X and μ is a fuzzy subset of a fuzzy topological space Y, then

(i) $pCl_C\,\lambda \times pCl_C\,\mu \geq pCl_C\,(\lambda \times \mu)$

(ii) $pInt_C\,\lambda \times pInt_C\,\mu \leq pInt_C\,(\lambda \times \mu)$.

Proof.

By using Definition 2.20, $(pCl_C\,\lambda \times pCl_C\,\mu)(x, y) = \min\{pCl_C\,\lambda(x), pCl_C\,\mu(y)\} \geq \min\{\lambda(x), \mu(y)\} = (\lambda\times\mu)(x, y)$. This shows that $pCl_C\,\lambda \times pCl_C\,\mu \geq (\lambda \times \mu)$.

By using Proposition 3.6, $pCl_C\,(\lambda \times \mu) \leq pCl_C\,(pCl_C\,\lambda \times pCl_C\,\mu) = pCl_C\,\lambda \times pCl_C\,\mu$.

By using Definition 2.10, $(pInt_C \lambda \times pInt_C \mu)(x, y) = \min\{pInt_C \lambda(x), pInt_C \mu(y)\} \leq \min\{\lambda(x), \mu(y)\} = (\lambda\times\mu)(x, y)$. This shows that $pInt_C \lambda \times pInt_C \mu \leq (\lambda \times \mu)$.

By using Proposition 3.2, $pInt_C (pInt_C \lambda \times pInt_C \mu) \leq pInt_C (\lambda \times \mu)$, that implies $pInt_C \lambda \times pInt_C \mu \leq pInt_C (\lambda \times \mu)$.

Theorem 4.23

Let X and Y be C -product related fuzzy topological spaces. Then for a fuzzy subset λ of X and a fuzzy subset μ of Y,

(i) $pCl_C (\lambda \times \mu) = pCl_C \lambda \times pCl_C \mu$

(ii) $pInt_C (\lambda \times \mu) = pInt_C \lambda \times pInt_C \mu$.

Proof.

By using Theorem 4.22, it is sufficient to show that $pCl_C (\lambda\times\mu) \geq pCl_C \lambda\times pCl_C \mu$. By using Definition 3.4, we have $pCl_C (\lambda \times \mu) = \inf\{ C (\lambda_\alpha \times \mu_\beta): C (\lambda_\alpha \times \mu_\beta) \geq \lambda \times \mu$ where λ_α and μ_β are fuzzy C - pre open$\}$. By using Lemma 2.12, we have

$pCl_C (\lambda \times \mu) = \inf \{ C \lambda_\alpha \times 1 \vee 1 \times C \lambda_\beta : C \lambda_\alpha \times 1 \vee 1 \times C \mu_\beta \geq \lambda \times \mu\}$

$= \inf \{ C \lambda_\alpha \times 1 \vee 1 \times C \mu_\beta : C \lambda_\alpha \geq \lambda$ or $C \mu_\beta \geq \mu\}$

$= \min (\inf \{ C \lambda_\alpha \times 1 \vee 1 \times C \mu_\beta : C \lambda_\alpha \geq \lambda\}, \inf\{ C \lambda_\alpha \times 1\vee 1 \times C \mu_\beta : C \mu_\beta \geq \})$.

Now $\inf \{ C \lambda_\alpha \times 1\vee 1 \times C \mu_\beta : C \lambda_\alpha \geq \lambda\} \geq \inf \{ C \lambda_\alpha \times 1 : C \lambda_\alpha \geq \lambda\}$

$= \inf \{C \lambda_\alpha : C \lambda_\alpha \geq \lambda\} \times 1$

$= (pCl_C \lambda) \times 1$.

Also $\inf \{ C \lambda_\alpha \times 1 \vee 1 \times C \mu_\beta : C \mu_\beta \geq \mu \} \geq \inf \{ 1 \times C \mu_\beta : C \mu_\beta \geq \mu \}$

$= 1 \times \inf \{C \mu_\beta : C \mu_\beta \geq \mu \}$

$= 1 \times pCl_C \mu$.

The above discussions imply that $pCl_C (\lambda \times\mu) \geq \min (pCl_C \lambda \times 1, 1 \times pCl_C \mu)$

$= pCl_C \lambda \times pCl_C \mu$.

(ii) follows from (i) and using Proposition 3.5.

Theorem 4.24

Let X_i, i = 1, 2 ….n, be a family of **C** -product related fuzzy topological spaces. If λ_i is a fuzzy subset of X_i, and the complement function **C** satisfies the monotonic and involutive conditions, then

$$pBd_C\left(\prod_{i=1}^{n}\lambda_i\right) = [pBd_C\,\lambda_1 \times pCl_C\,\lambda_2 \times \ldots \times pCl_C\,\lambda_n] \vee [pCl_C\,\lambda_1 \times pBd_C\,\lambda_2 \times \ldots \times pCl_C\,\lambda_n] \vee \ldots \vee [pCl_C\,\lambda_1 \times pCl_C\,\lambda_2 \times \ldots \times pBd_C\,\lambda_n].$$

Proof.

It suffices to prove this for n = 2. By using Proposition 4.14, we have $pBd_C(\lambda_1 \times \lambda_2) = pCl_C(\lambda_1 \times \lambda_2) \wedge C(pInt_C(\lambda_1 \times \lambda_2))$

$= (pCl_C\,\lambda_1 \times pCl_C\,\lambda_2) \wedge C(pInt_C\,\lambda_1 \times pInt_C\,\lambda_2)$ [by using Theorem 4.23]

$= (pCl_C\,\lambda_1 \times pCl_C\,\lambda_2) \wedge C\,[(pInt_C\,\lambda_1 \wedge pCl_C\,\lambda_1) \times (pInt_C\,\lambda_2 \wedge pCl_C\,\lambda_2)]$

$= (pCl_C\lambda_1 \times pCl_C\,\lambda_2) \wedge [C(pInt_C\,\lambda_1 \wedge pCl_C\,\lambda_1) \times 1 \vee 1 \times C(pInt_C\,\lambda_2 \wedge pCl_C\,\lambda_1)]$. [by Lemma 2.22].

Since **C** satisfies the monotonic and involutive conditions, by using Proposition 3.5(i), Proposition 3.5(i) and also by using Lemma 2.10,

$pBd_C(\lambda_1 \times \lambda_2)$

$= (pCl_C\,\lambda_1 \times pCl_C\,\lambda_2) \wedge [(pCl_C\,C\,\lambda_1 \vee pInt_C\,C\,\lambda_1) \times 1 \vee 1 \times (pCl_C\,C\,\lambda_2 \vee pInt_C\,C\,\lambda_2)]$

$= (pCl_C\,\lambda_1 \times pCl_C\,\lambda_2) \wedge [pCl_C\,C\,\lambda_1 \times 1 \vee 1 \times pCl_C\,C\,\lambda_2]$

$= [(pCl_C\,\lambda_1 \times pCl_C\,\lambda_2) \wedge (pCl_C\,(C\,\lambda_1) \times 1)] \vee [(pCl_C\,(\lambda_1) \times pCl_C\,\lambda_2) \wedge (1 \times pCl_C\,(C\,\lambda_2))]$.

Again by using Lemma 2.18, we get $pBd_C(\lambda_1 \times \lambda_2)$

$= [(pCl_C\,\lambda_1 \wedge pCl_C\,(C\,\lambda_1)) \times (1 \wedge pCl_C\,\lambda_2)] \vee [(pCl_C\,\lambda_2 \wedge pCl_C\,(C\,\lambda_2)) \times (1 \wedge pCl_C\,\lambda_1)]$

$= [(pCl_C\,(\lambda_1) \wedge pCl_C\,(C\,\lambda_1)) \times pCl_C\,(\lambda_2)] \vee [(pCl_C\,(\lambda_2) \wedge pCl_C\,(C\,\lambda_2)) \times pCl_C\,(\lambda_1)]$

$pBd_C(\lambda_1 \times \lambda_2) = [pBd_C\,(\lambda_1) \times pCl_C\,(\lambda_2)] \vee [pCl_C\,(\lambda_1) \times pBd_C\,(\lambda_2)]$.

Theorem 4.25

Let f: X→Y be a fuzzy continuous function. Suppose the complement function **C** satisfies the monotonic and involutive conditions. Then

$pBd_C\,(f^{-1}(\mu)) \leq f^{-1}(pBd_C\,(\mu))$, for any fuzzy subset μ in Y.

Proof.

Let f be a fuzzy continuous function and μ be a fuzzy subset in Y. By using Definition 4.1, we have $pBd_C\,(f^{-1}(\mu)) = pCl_C\,(f^{-1}(\mu)) \wedge pCl_C\,(C\,(f^{-1}(\mu))$. By using Lemma 2.19, $pBd_C\,(f^{-1}(\mu)) = pCl_C\,(f^{-1}(\mu)) \wedge pCl_C\,(f^{-1}(C\,\mu))$. Since f is fuzzy continuous and $f^{-1}(\mu) \leq f^{-1}(pCl_C\,(\mu))$, it follows that $pCl_C\,(f^{-1}(\mu)) \leq f^{-1}(pCl_C\,(\mu))$. This together with the above imply that $pBd_C\,(f^{-1}(\mu)) \leq f^{-1}(pCl_C\,(\mu)) \wedge f^{-1}(pCl_C\,(C\,\mu))$. By using Lemma 2.21, $pBd_C\,(f^{-1}(\mu)) \leq f^{-1}(pCl_C\,(\mu) \wedge pCl_C\,(C\,\mu))$. That is $pBd_C\,(f^{-1}(\mu)) \leq f^{-1}(pBd_C\,(\mu))$.

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